

On Large H -Intersecting Families

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Abstract

A family \mathcal{F} of graphs on a fixed set of n vertices is called *triangle-intersecting* if for any $G_1, G_2 \in \mathcal{F}$, the intersection $G_1 \cap G_2$ contains a triangle. More generally, for a fixed graph H , a family \mathcal{F} is *H -intersecting* if the intersection of any two graphs in \mathcal{F} contains a sub-graph isomorphic to H .

In [5], Ellis, Filmus and Friedgut proved a 36-year old conjecture of Simonovits and Sós stating that the maximal size of a triangle-intersecting family is $(1/8)2^{n(n-1)/2}$. Furthermore, they proved a p -biased generalization, stating that for any $p \leq 1/2$, we have $\mu_p(\mathcal{F}) \leq p^3$, where $\mu_p(\mathcal{F})$ is the probability that the random graph $G(n, p)$ belongs to \mathcal{F} .

In the same paper, Ellis et al. conjectured that the assertion of their biased theorem holds also for $1/2 < p \leq 3/4$, and more generally, that for any non- t -colorable graph H and any H -intersecting family \mathcal{F} , we have $\mu_p(\mathcal{F}) \leq p^{t(t+1)/2}$ for all $p \leq (2t-1)/(2t)$.

In this note we construct, for any fixed H and any $p > 1/2$, an H -intersecting family \mathcal{F} of graphs such that $\mu_p(\mathcal{F}) \geq 1 - e^{-n^2/C}$, where C depends only on H and p , thus disproving both conjectures.

1 Introduction

Denote $[n] = \{1, 2, \dots, n\}$. Throughout the paper, \mathcal{G}_n denotes the family of all graphs on a fixed set of n vertices.

A family $\mathcal{F} \subset \mathcal{P}([n])$ is said to be *intersecting* if for any $A, B \in \mathcal{F}$, $A \cap B \neq \emptyset$. The classical Erdős-Ko-Rado (EKR) theorem [7] determines the maximal size of an intersecting family of k -element subsets of $[n]$.

Theorem 1.1 (Erdős, Ko, and Rado, 1961). *Let $k < n/2$, and let $\mathcal{F} \subset [n]^{(k)}$ be an intersecting family. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. Equality holds if and only if $\mathcal{F} = \{A \in [n]^{(k)} : x \in A\}$, for some $x \in [n]$.*

The EKR theorem is the cornerstone of an entire subfield of extremal combinatorics called ‘intersection problems for finite sets’, which studies how large can a family of sets be given some intersection constraints on its elements. See [9] for a recent survey of the topic.

Along with intersection problems on families of k -element sets (called k -uniform families), it is quite common to consider p -biased versions of the problems, in which one wants to find the maximal p -biased measure of a family $\mathcal{F} \subseteq \mathcal{P}([n])$, defined by $\mu_p(\mathcal{F}) := \sum_{S \in \mathcal{F}} p^{|S|}(1-p)^{n-|S|}$,

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given that \mathcal{F} satisfies some intersection constraints. In this setting, the basic result is the p -biased version of the EKR Theorem, proved by Ahlswede and Katona [1] in 1977, which asserts that for any intersecting \mathcal{F} and any $p \leq \frac{1}{2}$ we have $\mu_p(\mathcal{F}) \leq p$. Biased intersection theorems usually follow from the corresponding k -uniform results (see [4]). In the other direction, in some cases k -uniform results were deduced from their p -biased analogues (see, e.g., [10, 11]). For a semi-random sample of recent p -biased intersection results, see [6, 8, 11] and the references therein.

One of the best-known intersection problems was determining the maximal size of a triangle-intersecting family of graphs. In 1976, Simonovits and Sós conjectured that the maximum is attained by the family of all graphs that contain a fixed triangle, and thus, $|\mathcal{F}| \leq (1/8)2^{n(n-1)/2}$ for any triangle-intersecting $\mathcal{F} \subset \mathcal{G}_n$. The first major step toward resolution of the conjecture was made in 1986, when Chung et al. [3] showed that $|\mathcal{F}| \leq (1/4)2^{n(n-1)/2}$ using entropy methods. It took 26 more years until the Simonovits-Sós conjecture was proved in a beautiful paper of Ellis, Filmus and Friedgut [5] using spectral methods and Fourier analysis. Actually, Ellis et al. proved the following more general biased version of the conjecture:

Theorem 1.2 (Ellis, Filmus, and Friedgut, 2012). *Let \mathcal{F} be a triangle-intersecting family of graphs. Then for any $p \leq 1/2$, we have $\mu_p(\mathcal{F}) \leq p^3$.*

(Note that the Simonovits-Sós conjecture is the case $p = 1/2$ of Theorem 1.2.) Ellis et al. also proved several extensions and variants of the theorem, including a version for odd-cycle intersecting families and a k -uniform version for any $k = \alpha \binom{n}{2}$, $\alpha < 1/2$.

In the same paper, Ellis et al. conjectured that Theorem 1.2 holds in much larger generality.

Conjecture 1.3 ([5]). *Let H be a non- t -colorable graph. Then for any H -intersecting family \mathcal{F} and any $p \leq \frac{2t-1}{2t}$, we have*

$$\mu_p(\mathcal{F}) \leq p^{\binom{t+1}{2}},$$

and the maximum is attained if and only if H is the complete graph on $t+1$ vertices.

Note that one can easily show, using the classical Turán's theorem, that for $p > \frac{2t-1}{2t}$ there exist K_{t+1} -intersecting families of measure $1 - o(1)$. Thus, Conjecture 1.3 is the strongest result one may hope for.

In this note we prove the following result, which disproves Conjecture 1.3.

Theorem 1.4. *For any graph H and any $p > \frac{1}{2}$, there exists an H -intersecting graph family $\mathcal{F} \subset \mathcal{G}_n$ such that $\mu_p(\mathcal{F}) \geq 1 - e^{-n^2/C}$, where $C = C(p, H) > 0$.*

The proof of Theorem 1.4 is rather simple – we construct inductively a sequence of families $\{\mathcal{F}_t\}_{t=2,3,\dots}$, such that each \mathcal{F}_t is K_t -intersecting, and show that $\mu_p(\mathcal{F}_t)$ satisfies the assertion of the theorem using classical Chernoff bounds [2].

Note that by the biased EKR theorem mentioned above, for any non-empty H and any $p \leq 1/2$, any H -intersecting family \mathcal{F} satisfies $\mu_p(\mathcal{F}) \leq p$ (and in particular, there do not exist H -intersecting families of p -measure close to 1). Hence, our result implies that the maximal p -measure of an H -intersecting family exhibits a *sharp threshold* phenomenon at $p = 1/2$. It may be interesting to further understand the ‘threshold window’, and in particular, to determine the maximal $p = p(n)$ such that for any H -intersecting family \mathcal{F} , $\mu_p(\mathcal{F})$ is bounded away from 1 (for some fixed graph H).

2 Proof of Theorem 1.4

We use the following standard consequence of Chernoff's inequality (see [2], Appendix A).

Proposition 2.1. *For any $N \geq 1$ and $p' < p < 1$, there exists a constant $C = C(p, p') > 0$ such that the following holds. Let $X \sim \text{Bin}(N, p)$. Then*

$$\Pr[X \leq p'] \leq e^{-N/C}.$$

Proof of Theorem 1.4. It is clearly sufficient to prove the theorem for all complete graphs $H = K_t$, $t \in \mathbb{N}$. We prove the theorem by induction on t , by constructing (for each t) a K_t -intersecting family $\mathcal{F}_t^n \subset \mathcal{G}_n$ that satisfies the assertion of the theorem. Recall that for any family $\mathcal{F} \subset \mathcal{G}_n$, and for any p , $\mu_p(\mathcal{F})$ is the probability that a random graph $G \sim G(n, p)$ belongs to \mathcal{F} .

For $t = 2$ and for any $n \in \mathbb{N}$, we define $\mathcal{F}_2^n \subset \mathcal{G}_n$ as the family of all graphs that contain more than half of the $\binom{n}{2}$ possible edges. \mathcal{F}_2 is clearly K_2 -intersecting, and by Proposition 2.1, we have

$$\mu_p(\mathcal{F}_2^n) \geq 1 - e^{-n^2/C},$$

where $C = C(p)$, as asserted. (Note that the number of edges in $G \sim G(n, p)$ has distribution $\text{Bin}(\binom{n}{2}, p)$, and thus we indeed can apply Proposition 2.1 to bound $\Pr_{G \sim G(n, p)}[G \in \mathcal{F}_2^n]$.)

Suppose that we already defined K_t -intersecting families $\mathcal{F}_t^m \subset \mathcal{G}_m$ (for all $m \in \mathbb{N}$) such that $\mu_p(\mathcal{F}_t^m) \geq 1 - e^{-m^2/C(p, t)}$. For any $n \in \mathbb{N}$, we define $\mathcal{F}_{t+1}^n \subset \mathcal{G}_n$ to be the family of all graphs G such that:

1. G has at least $\frac{p+0.5}{2}\binom{n}{2}$ edges.
2. For every subset $S \subseteq [n]$ with $|S| \geq (p - \frac{1}{2})(n - 1)$, the induced sub-graph of G on the vertex set S (denoted by $G|_S$) belongs to $\mathcal{F}_t^{|S|}$.

Let $G \sim G(n, p)$. By Proposition 2.1, we have

$$\Pr[G \text{ satisfies (1)}] \geq 1 - e^{-n^2/C(p)}.$$

In addition, for any fixed $S \subset [n]$ with $|S| \geq (p - 0.5)(n - 1)$, we have

$$\Pr[G|_S \in \mathcal{F}_t^{|S|}] \geq 1 - e^{-(p-0.5)^2(n-1)^2/C(p, t)}$$

by the induction hypothesis. Hence, a union bound implies

$$\mu_p(\mathcal{F}_{t+1}^n) \geq 1 - e^{-n^2/C(p)} - \sum_{S \subseteq [n]} e^{-(p-0.5)^2(n-1)^2/C(p, t)} \geq 1 - e^{-n^2/C(t+1, p)}.$$

We assert that \mathcal{F}_{t+1}^n is K_{t+1} -intersecting. To prove this, let $G_1, G_2 \in \mathcal{F}_{t+1}^n$, and let $G_0 = G_1 \cap G_2$. We show that G_0 contains a copy of K_{t+1} .

Let v be a vertex of maximal degree in G_0 . As $|E(G_0)| \geq |E(G_1)| + |E(G_2)| - \binom{n}{2} \geq (p - 0.5)\binom{n}{2}$, we have $\deg(v) \geq (p - 0.5)(n - 1)$. Let T be the set of neighbors of v in G_0 , and note that $|T| \geq (p - 0.5)(n - 1)$. It is clearly sufficient to show that the induced sub-graph $(G_0)|_T$ contains a copy of K_t . Consider the induced sub-graphs $(G_1)|_T, (G_2)|_T$, and $(G_0)|_T$. By assumption, we have $(G_1)|_T, (G_2)|_T \in \mathcal{F}_t^{|T|}$. Since $\mathcal{F}_t^{|T|}$ is K_t -intersecting, this implies that $(G_0)|_T = (G_1)|_T \cap (G_2)|_T$ contains a copy of K_t . This completes the proof. \square

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